Exercise 10

Solve the differential equation.

$$\frac{d^2y}{dx^2} + y = \csc x, \quad 0 < x < \pi/2$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$\frac{d^2y_c}{dx^2} + y_c = 0 \tag{1}$$

This is a linear homogeneous ODE with constant coefficients, so it has solutions of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad \frac{dy_c}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y_c}{dx^2} = r^2e^{rx}$$

Substitute these formulas into equation (1).

$$r^2e^{rx} + e^{rx} = 0$$

Divide both sides by e^{rx}.

$$r^2 + 1 = 0$$

Solve for r.

$$r = \{-i, i\}$$

Two solutions to the ODE are e^{-ix} and e^{ix}. According to the principle of superposition, the general solution to equation (1) is a linear combination of these two.

$$\begin{aligned} y_c(x) &= C_1e^{-ix} + C_2e^{ix} \\ &= C_1(\cos x - i\sin x) + C_2(\cos x + i\sin x) \\ &= (C_1 + C_2)\cos x + (-iC_1 + iC_2)\sin x \\ &= C_3\cos x + C_4\sin x \end{aligned}$$

C_3 and C_4 are arbitrary constants. On the other hand, the particular solution satisfies the original ODE.

$$\frac{d^2y_p}{dx^2} + y_p = \csc x \tag{3}$$

Use the method of variation of parameters to obtain it: Allow the parameters in the complementary solution to vary.

$$y_p(x) = C_3(x)\cos x + C_4(x)\sin x$$

Differentiate it with respect to x.

$$\frac{dy_p}{dx} = C_3'(x)\cos x + C_4'(x)\sin x - C_3(x)\sin x + C_4(x)\cos x$$

If we set

$$C_3'(x)\cos x + C_4'(x)\sin x = 0, \tag{4}$$

then

$$\frac{dy_p}{dx} = -C_3(x)\sin x + C_4(x)\cos x.$$

Differentiate it with respect to x once more.

$$\frac{d^2y_p}{dx^2} = -C_3'(x)\sin x + C_4'(x)\cos x - C_3(x)\cos x - C_4(x)\sin x$$

Substitute these formulas into equation (3).

$$[-C_3'(x)\sin x + C_4'(x)\cos x - \cancel{C_3(x)\cos x} - \cancel{C_4(x)\sin x}] + [\cancel{C_3(x)\cos x} + \cancel{C_4(x)\sin x}] = \csc x$$

Simplify the left side.

$$-C_3'(x)\sin x + C_4'(x)\cos x = \csc x \tag{5}$$

Multiply both sides of equation (3) by $\sin x$, and multiply both sides of equation (4) by $\cos x$.

$$C_3'(x)\cos x\sin x + C_4'(x)\sin^2 x = 0$$

$$-C_3'(x)\cos x\sin x + C_4'(x)\cos^2 x = \cot x$$

Add the respective sides of these equations.

$$C_4'(x) = \cot x$$

Integrate both sides to get $C_4(x)$, setting the integration constant to zero.

$$C_4(x) = \ln|\sin x|$$

Note that because $0 < x < \pi/2$, the absolute value sign can be dropped.

$$C_4(x) = \ln(\sin x)$$

Solve equation (4) for $C_3'(x)$.

$$\begin{aligned} C_3'(x) &= -C_4'(x)\frac{\sin x}{\cos x} \\ &= -(\cot x)\frac{\sin x}{\cos x} \\ &= -1 \end{aligned}$$

Integrate both sides to get $C_3(x)$, setting the integration constant to zero.

$$C_3(x) = -x$$

The particular solution is then

$$\begin{aligned} y_p &= C_3(x)\cos x + C_4(x)\sin x \\ &= (-x)\cos x + [\ln(\sin x)]\sin x \\ &= \ln(\sin x)\sin x - x\cos x. \end{aligned}$$

Therefore, the general solution to the original ODE is

$$\begin{aligned} y &= y_c + y_p \\ &= C_3\cos x + C_4\sin x + \ln(\sin x)\sin x - x\cos x. \end{aligned}$$